A BAND-LIMITED SIGNAL EXTRAPOLATION ALGORITHM USING PSEUDO INVERSE FILTERING AND HEURISTIC OPTIMAZATION

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ABSTRACT A band-limited signal extrapolation algorithm is proposed. First, Cadzow's algorithm is modified to eliminate undesired outband spectra. Next, an inverse filter stopband response is relaxed by adding small random numbers. A constrained heuristic optimization is suggested, which can employ arbitrary signal properties as constraints. Through numerical examples, these regularization techniques can improve SNR by 20 dB.

I. INTRODUCTION

The extrapolation of a band-limited signal in terms of a finite segment is one of the most important problems in signal processing. Various algorithms have been proposed for this problem.

They can be clasified into the following categories. First is the analytical continuation technique, and the series expansion technique in terms of prolate spheroidal functions [1]. The second category is an iterative algorithm, based on successive reduction of the mean-square error [2]-[5]. In the author's experience, this algorithm works well only when the frequency reponse for a finite segment is similar to the original.

The third algorithm is a one-step procedure [6],[7]. In Cadzow's algorithm [7], the original signal is formulated as a convolution sum of an unknown signal and a band-limiting filter. Since they are sampled by a much higher frequency than the original signal bandwidth, undesired outband spectra can be generated, which easily disperses an extrapolation process.

The band-limited signal extrapolation is basically an ill-posed problem. In actuality, $x_{PT}(n)$ contains noise, and errors associated with algorithm implementation can exist. In order to regularize an extrapolation process, additional knowledge about $x(n)$ beyond its band-limited character should be taken into account. Energy constraints for the band-limited signal or for the noise have been used for this purpose [8]-[11].

This paper also deals with regularization techniques for ill-posed inverse filtering . First, Cadzow's algorithm is modified to eliminate undesired outband spectra. Second, in order to suppress noise amplification, an inverse filter stopband response is relaxed. Third, a constrained heuristic optimization is suggested. Finally, numerical examples are demonstrated.

II. MODIFIED CADZOW'S ALGORITHM

Cadzow's Algorithm

The original signal $x(n)$ is assumed to be

$$x(n)=g(n)*u(n),\ n\in\Lambda \qquad (1)$$

$u(n)$ is an appropriate input signal. $g(n)$ is an impulse response of a band-limiting filter. Symbol $*$ means a convolution sum. Λ is a set of sampling points located in the entire time interval from n=0, to N-1.

Based on the above assumption, an extrapolation procedure is formulated as

$$x_{PT}(n)=h(n)*\hat{u}(n),\ n\in\Lambda_{PT} \qquad (2)$$

$$\hat{x}(n)=h(n)*\hat{u}(n),\ n\in\Lambda \qquad (3)$$

$x_{PT}(n)$ is a finite segment, that is a fractionally observed data. Λ_{PT} is a set of sampling points in the observation intervals. $h(n)$ is an impulse response of a band-limiting filter, used in an extrapolation process. $\hat{u}(n)$ and $\hat{x}(n)$ are estimated versions of $u(n)$ and $x(n)$, respectively.

In order to obtain $\hat{u}(n)$ by solving Eq.(2), the number of $x_{PT}(n)$ should be equal to or larger than that of $\hat{u}(n)$ For this purpose, $x_{PT}(n)$ is oversampled in a narrow time interval. In Cadzow's method, $\hat{u}(n)$ is also sampled by the high frequency. As a result, $\hat{u}(n)$ contains undesired outband spectra, which easily cause unbounded error.

Modified Algorithm

The original signal $x(n)$ is assumed to be bandlimited within fc. Since $x_{PT}(n)$ should be oversampled in a narrow interval, $x(n)$ is expressed with a much higher sampling frequency than 2fc. Let fs be the sampling frequency, satisfying $fs \gg fc$. The frequency response for $x(n)$ in $0\leq f\leq fc$ does not change by sampling $x(n)$ with 2fc. Let $x_D(nL)$ be the decimated version of $x(n)$ by 2fc/fs, as follows:

$$x_D(nL)=x(nL) \qquad (4a)$$

$$x_D(m)=0,\ m\neq nL \qquad (4b)$$

$$L=fs/2fc \qquad (4c)$$

Althogh $x_D(nL)$ is sampled with 2fc, it is represented as $x_D(n)$ for convenience. This means L-1 samples among L samples are zero.

The original signal $x(n)$ can be reconstructed by eliminating the outband spectra for $x_D(n)$.

$$x(n)=h(n)*x_D(n) \tag{5}$$

A cutoff frequency for $h(n)$ is set to be fc. This process is illustrated in Fig.1. Since $x(n)$ is a finite sequence,

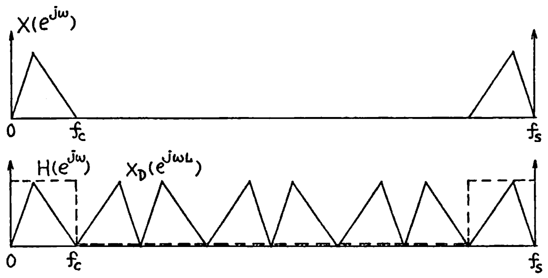


Fig.1 $x(n)$ is obtained by passing $x_D(nL)$ through band-limiting filter.

the band-limitation by $H(z)$ is not complete. Therefore, Eq.(5) is approximately held.

Let $\hat{x}_D(n)$ be the approximated version of $x_D(n)$. It is obtained from

$$x_{PT}(n)=h(n)*\hat{x}_D(n),\ n\in\Lambda_{PT} \tag{6}$$

The entire extrapolated signal $\hat{x}(n)$ is obtained by interpolating $\hat{x}_D(n)$, as follows:

$$\hat{x}(n)=h(n)*\hat{x}_D(n),\ n\in\Lambda \tag{7}$$

The advantage of the proposed approach over Cadzow's method is elimination of the undesired outband spectra.

<u>Number of $\hat{x}_D(n)$ Samples</u>

Letting M be the number of $\hat{x}_D(n)$ samples, it is determined from

$$M=N(2fc/fs) \tag{8}$$

If M is given by

$$M=mN(2fc/fs),\ 1<m:\text{integer} \tag{9}$$

its sampling frequency becomes 2mfc. This causes the undesired spectra in the outband $fc\leqq f\leqq mfc$, which disperses an extrapolation process.

On the other hand, when M is smaller than Eq.(8), the sampling frequency is less than 2fc. Therefore, $\hat{x}_D(n)$ cannot represent the entire frequency response. However, by assuming that $\hat{x}_D(n)$ is equally spaced in a partial range of Λ, the sampling frequency can be set to 2fc,

<u>Number of $x_{PT}(n)$ Samples</u>

Letting K be the number of $x_{PT}(n)$, it should satisfy

$$K\geqq M, \tag{10}$$

in order to obtain $\hat{x}_D(n)$ from Eq.(6). When K=M, Eq.(6) becomes a set of linear equations for $\hat{x}_D(n)$. On the other hand, when K is larger than M, $\hat{x}_D(n)$ is obtained as the least mean square solution from Eq.(6). In both ways, estimating $\hat{x}_D(n)$ is carried out by solving linear equations.

III. PSEUDO INVERSE FILTERING

<u>Noise Amplification</u>

Equation (6) is expressed using vectors and a matrix, as follows:

$$H\underline{\hat{x}}_D=\underline{x}_{PT} \tag{11}$$

H is an MxM matrix, which consists of $h(n)$. $\underline{\hat{x}}_D$ and $\underline{x}_{PT}$ are vectors for $\hat{x}_D(n)$ and $x_{PT}(n)$, respectively. From the above equation, $\underline{\hat{x}}_D$ is calculated by

$$\underline{\hat{x}}_D=H^{-1}\underline{x}_{PT} \tag{12}$$

Next, noise $\varepsilon_{PT}(n)$ in $x_{PT}(n)$ is taken into account. Equation (11) is rewritten as

$$H\underline{\hat{x}}_D=\underline{x}_{PT}+\underline{\varepsilon}_{PT} \tag{13}$$

$\underline{\hat{x}}_D$ is now expressed by

$$\underline{\hat{x}}_D=H^{-1}\underline{x}_{PT}+H^{-1}\underline{\varepsilon}_{PT} \tag{14}$$

The second term in the right hand side is noise after signal extrapolation. Thus, the noise is amplified by H^{-1}. Althogh H^{-1} is determined by Λ_{PT}, it can be regarded as inverse filtering. Therefore, noise spectra in the stopband are greatly amplified.

<u>Adding Small Random Numbers</u>

In order to suppress this noise amplification, the stopband amplitude response is relaxed by adding small random numbers to $h(n)$.

$$h'(n)=h(n)+\lambda r(n) \tag{15}$$

$r(n)$ represents random numbers distributed in the range (-0.5,0.5), and λ is a scaling factor. From Eq.(15), Eq.(6) is rewritten as

$$x_{PT}(n)=(h(n)+\lambda r(n))*\hat{x}_D(n) \tag{16}$$

Furthermore, Eq.(13) becomes

$$(H+\lambda R)\underline{\hat{x}}_D=\underline{x}_{PT}+\underline{\varepsilon}_{PT} \tag{17}$$

R is an MxM matrix, which consists of $\lambda r(n)$. From this equation, $\underline{\hat{x}}_D$ is obtained as

$$\begin{aligned}\underline{\hat{x}}_D&=(H+\lambda R)^{-1}(\underline{x}_{PT}+\underline{\varepsilon}_{PT})\\&=(H+\lambda R)^{-1}\underline{x}_{PT}+(H+\lambda R)^{-1}\underline{\varepsilon}_{PT}\end{aligned} \tag{18}$$

After $\underline{\hat{x}}_D$ is obtained, the entire signal $\hat{x}(n)$ is interpolated through Eq.(7), which does not include the random numbers.

Letting $H'(z)$ be the z transform of $h'(n)$, they are related by Parseval's relation.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H'(e^{jw})|^2\,d\omega=\sum_{n=0}^{N_h-1}h'^2(n) \tag{19}$$

N_h is the number of $h(n)$ samples. Since $h(n)$ and $r(n)$ are independent from each other, the $h'(n)$ energy is divided into

$$\sum_{n=0}^{N_h-1}h'^2(n)=\sum_{n=0}^{N_h-1}h^2(n)\ +\ \lambda^2\sum_{n=0}^{N_h-1}r^2(n) \tag{20}$$

Letting $H(z)$ and $R(z)$ be the z transforms of $h(n)$ and $r(n)$, respectively, Eq.(20) is translated into a frequency domain, as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H'(e^{jw})|^2d\omega=$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega + \frac{\lambda^2}{2\pi}\int_{-\pi}^{\pi}|R(e^{j\omega})|^2 d\omega \quad (21)$$

Since $|R(e^{j\omega})|$ can be assumed to be flat in the entire frequency band, the second term in the right hand side becomes

$$\lambda^2 |R(e^{j\omega})|^2 = \lambda^2 \sum_{n=0}^{N_h-1} r^2(n) \quad (22)$$

When $r(n)$ has zero mean value, and is uniformly distributed in (-0.5,0.5), its spectrum can be evaluated by

$$\lambda^2 |R(e^{j\omega})|^2 = N_h \lambda^2/12 \quad (23)$$

Since the stopband amplitude response of $H'(z)$ is mostly determined by $\lambda R(z)$, inverse matrix stability can be controlled by λ. The passband response is almost the same as that for $H(z)$, because λ is usually a very small number.

On the other hand, $x_D\hat{}(n)$ cannot be completely band-limited, due to λR included in the first term in the right hand side of Eq.(17). This degrades an extrapolated signal. Thus, the optimum trade-off for λ should be determined, taking both effects into account.

IV. HEURISTIC OPTIMIZATION METHOD

A constrained heuristic optimization procedure is suggested as another regularization technique. It makes it possible to use arbitrary signal properties as constraints.

Let $x_{PT}'(n)$ be the observed data, including $\varepsilon_{PT}(n)$.

$$x_{PT}'(n) = x_{PT}(n) + \varepsilon_{PT}(n) \quad (24)$$

If we assume $|\varepsilon_{PT}(n)| < \delta$, the exact $x_{PT}(n)$ is expected within $x_{PT}'(n) \pm \delta$.

In order to minimize the distance between $x_{PT}(n)$ and $x_{PT}'(n)$, a supplementary sequence $\Delta(n)$ is added to $x_{PT}'(n)$The optimum $\Delta(n)$ is searched for through a heuristic procedure, based on some error criterion.

Letting $\Gamma[x(n)]$ represent observable property for $x(n)$, an error criterion is, for example, defined by

$$E = |\Gamma[x(n)] - \Gamma[x\hat{}(n)]| / \Gamma[x(n)] \quad (25)$$

Equation (11) is rewritten, taking $\Delta(n)$ into account.

$$H\underline{x}_D\hat{} = \underline{x}_{PT}' + \underline{\Delta} \quad (26)$$

After $\underline{x}_D\hat{}$ is obtained from the above equation, $x\hat{}(n)$ is calculated from Eq.(7).

First, in the following region,

$$-\delta \leqq \Delta(n) \leqq \delta \quad (27)$$

Q sets of random numbers are assigned to $\Delta(n)$. From Eqs.(26), (7) and (25), E is calculated for each $\Delta(n)$ set. Among them, the optimum $\Delta(n)$ set is selected, which minimizes E. Letting $\underline{\Delta}^{(1)}$ be the optimum set on the first stage, a more optimum set is further searched for around $\underline{\Delta}^{(1)}$ on the second stage. The search region is narrowed, stage by stage. By repeating this process on the several stages, quasi-optimum $\Delta(n)$ is obtained.

Since $\varepsilon_{PT}(n)$ is regarded as a random sequence, and the noise power is roughly estimated, the proposed heuristic procedure can find a quasi-optimum solution in a short computing time.

V. NUMERICAL EXAMPLES

The original signal $x(n)$ is given by

$$x(n) = g(n) * v(n) \quad (28)$$

A cutoff frequency for $g(n)$ is $fc=fs/16$. $v(n)$ is a multi-frequency signal given by

$$v(n) = \sin(\omega_0) + \sin(2\omega_0) + \sin(3\omega_0) \quad (29a)$$

$$\omega_0 = 2\pi/64 \quad (29b)$$

The number of $x(n)$ samples is N=128, and $g(n)$ and $v(n)$ samples numbers are both N/2=64. $x_D\hat{}(n)$ is assumed to be sampled by 2fc in $0 \leqq n \leqq N/2-1$. The number of $x_D\hat{}(n)$ is (N/2)(2fc/fs)=64. Seven samples among every eight samples are set to be zero. A signal-to-noise ratio is defined by

$$SNR = 10\log\{E[(x(n)-x\hat{}(n))^2]/E[x^2(n)]\} \ [dB] \quad (30)$$

$E[x(n)]$ is a mean value for $x(n)$.

$x(n)$, $x_{PT}(n)$ and their amplitude responses are shown in Figs.2(a) and 2(b), respectively. Λ_{PT} is from n=61 to 68. $x(n)$ is normalized so that the muximum value is unity and quantized with 20 bit wordlengths. Extrapolated results are shown in Fig.2(c). Second, the normalized $x(n)$ is quantizedwith 12 bit wordlengths. As Fig.2(d) shows, the extrapolated results significantly deviate from the original. Since all graphs are normalized, so that their maximum value have the same value, $x\hat{}(n)$ in Λ_{PT} is slightly different from $x(n)$.

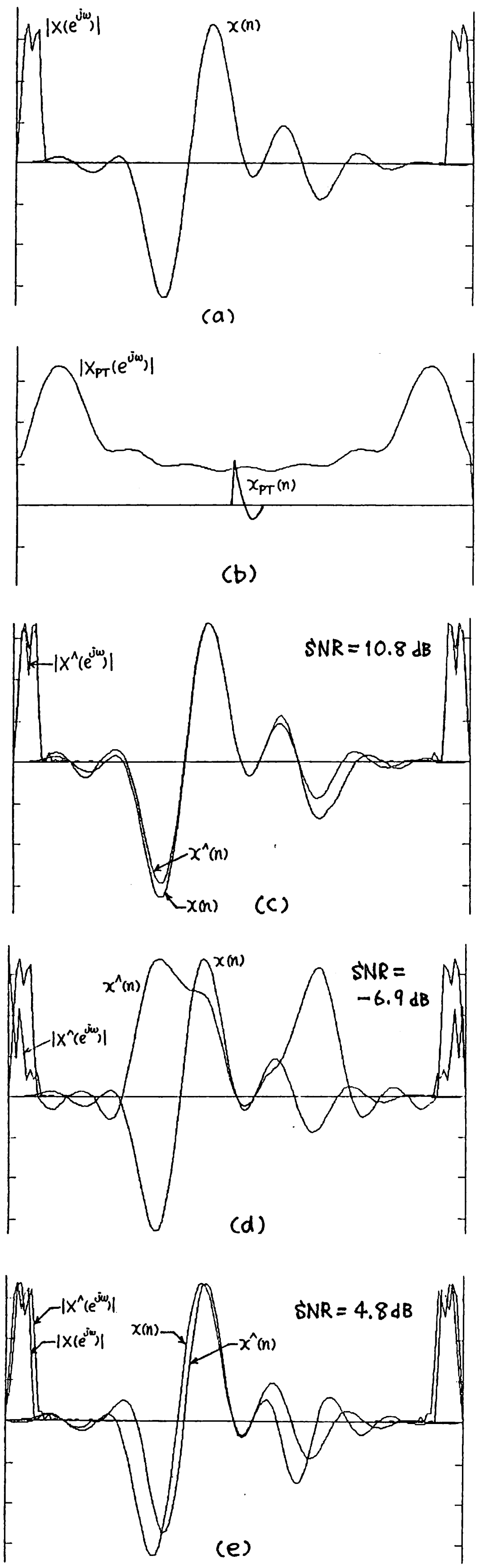
Next, small random numbers (λ=0.005) are added to $h(n)$. As Fig.2(e) shows. the results are more like the original. SNR is increased by 11.7 dB. Furthermore, the heuristic optimization was carried out, using energy and variance for $x(n)$ as constraints. The variance is defined by

$$Var[x(n)] = \sum_{n=0}^{N-1} |N/2-n| x^2(n) \quad (31)$$

Fifty sets of random numbers are assigned to $\Delta(n)$ on one stage. The searching process is repeated on three stages, narrowing the search region by 1/4. The results are shown in Fig.2(f). SNR is further improved by 10 dB from the psuedo inverse filtering.

VI. CONCLUSION

A band-limited signal extrapolation algorithm has been proposed. First, Cadzow's algorithm has been modified to eliminate outband spectra. Next, two kinds of regularization techniques have been proposed. They are inverse filter relaxation and constrained heuristic optimization. Through numerical examples, these regularization techniques improved SNR by 20 dB.



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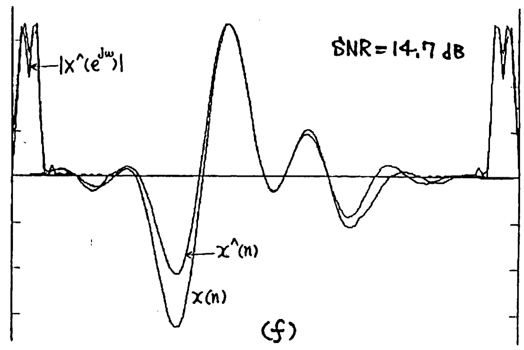

Fig.2 Numerical examples for proposed signal extrapolation algorith. (a) $x(n)$ and its amplitude response. (b) $x_{PT}(n)$ and its amplitude response. Extrapolated signals and their amplitude responses are shown in (c)~(f). (c) $x_{PT}(n)$ is quantized with 20 bit wordlengths. (d) $x_{PT}(n)$ is quantized with 12 bit wordlengths. (e) Pseudo inverse filtering, $\lambda=0.005$. (f) Constrained heuristic optimization.